DIFFERENTIALS OF THE SECOND KIND ON MUMFORD CURVES

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ABSTRACT

We define a p-adic analytic Hodge decomposition for the cohomology of Mumford curves, with values in a local system. When the local system is trivial, we give a new proof of Gerritzen's theorem, that this decomposition forms a variation of Hodge structure, in a family of Mumford curves.

Our purpose is to study differentials of the first and second kind *with values in a local system* on Mumford curves. In particular we look at the p-adic analytic version of the Hodge decomposition.

When the local system is trivial, the results of this paper follow from [Ge2],[Ge3]. However, the explicit construction of differentials of the first kind as logarithmic derivatives of theta functions (due to Manin and Drinfel'd) doesn't carry over to non-trivial local systems. Neither does the construction of certain differentials of the second kind, as in the first paper cited above. Our approach is therefore different than Gerritzen's. It avoids explicit formulae or theta functions, and is based instead on [dS1] and on Coleman's theory of integration [C] (although very little of the latter is actually needed here).

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§1. Hodge decomposition on Mumford curves

1.1. With notation as in [dS1] and [dS2], let K be a finite extension of $\mathbf{Q}_p$ and Γ a discrete finitely generated subgroup of $\mathrm{SL}_2(K)$. For simplicity, although this

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is not essential, we assume that Γ is torsion-free and Zariski dense. The first assumption means that Γ is a p-adic Schottky group [Ge–vdP]. According to Ihara's theorem Γ is then free, say of rank g. The second assumption is equivalent to $g \geq 2$, because any proper algebraic subgroup of SL_2 is contained in a Borel subgroup B, and $B(K)$ does not contain non-cyclic discrete groups.

The limit set $\mathfrak{L}_\Gamma$ of Γ is a compact subset of $\mathbf{P}^1(K)$, and its complement in $\mathbf{P}^1$ is a rigid analytic space $\mathfrak{h}_\Gamma = \mathbf{P}^1 - \mathfrak{L}_\Gamma$ defined over K, on which Γ acts discontinuously. $\Gamma\backslash\mathfrak{h}_\Gamma$ is the rigid analytic space associated with a unique smooth complete curve X_Γ over K (we shall abbreviate X_Γ by X). Curves admitting such a uniformization are called Mumford curves.

PROPOSITION (fundamental domain). *There exist* $\gamma_1, \ldots, \gamma_g$ *in* Γ, $\alpha_1, \ldots, \alpha_g$ *in* $GL_2(K)$, *and* $\mu_1, \ldots, \mu_g$ *in* $K^\times$ *with the following properties.*

(i) *The* γ_i *freely generate* Γ.

(ii) $\alpha_i \circ \gamma_i \circ \alpha_i^{-1}(z) = \mu_i z$, *and* $|\mu_i| < 1$.

(iii) *If* z_j^+ (*resp.* z_j^-) *denotes the attractive* (*resp. repulsive*) *fixed point of* γ_j, *then* $|\mu_i| < |\alpha_i(z_j^\pm)| \leq 1$ *for* $i \neq j$.

PROOF. Let $\gamma_1, \ldots, \gamma_g$ be a Schottky basis for Γ and $w_i(z) = (z - z_i^+)/(z - z_i^-)$. Then $w_i \circ \gamma_i \circ w_i^{-1}(z) = \mu_i z$, $|\mu_i| < 1$, and for $j, k \neq i$ ([Ge1], §2)

$$|\mu_i| < |w_i(z_j^\pm)/w_i(z_k^\pm)| < |\mu_i|^{-1}$$

(the choices of $\pm$ in the numerator and denominator are independent of each other). Choose c_i so that $|c_i| = \max_{j \neq i} |w_i(z_j^\pm)|$, and let $\alpha_i(z) = c_i^{-1} w_i(z)$. The proposition follows. ∎

Notice that the same α_i, γ_i and μ_i are also good for any finite extension of K. For the next definition, however, we fix a uniformizer π of K, and set

$$a_i = \{z \,\big|\, |\pi^{-1}\mu_i| \leq |\alpha_i(z)| \leq 1\},$$

$$b_i = \{z \,\big|\, 1 < |\alpha_i(z)| < |\pi^{-1}|\},$$

$$c_i = \{z \,\big|\, |\mu_i| < |\alpha_i(z)| < |\pi^{-1}\mu_i|\}.$$

Then $c_i = \gamma_i(b_i)$ and, if $j \neq i$, b_j and c_j are contained in a_i. If we set $W_i = a_i \cup c_i$, then $W = \bigcap W_i$ is a *fundamental domain* for $\Gamma\backslash\mathfrak{h}_\Gamma$ which is independent of K. W is the complement in the projective line of $2g$ K-rational disks, g of them closed and g open.

We orient the annuli b_i and c_i as in [dS1] or [dS2]: b_i points into a_i, and c_i out of it.

1.2. Let $V_n = \mathrm{Symm}^n(st)$ be the nth symmetric power of the standard representation of $\mathrm{SL}_2(K)$, and $\langle . , . \rangle$ the invariant bilinear pairing on it ([dS1]3.6.2). V_n gives rise to a local system of K-vector spaces $\boldsymbol{V}_n$ on X (i.e. a locally constant sheaf of vector spaces in the rigid analytic topology). If pr denotes the projection $\mathfrak{h}_\Gamma \to X$, and U is an admissible open set of X, then $\boldsymbol{V}_n(U) = \{f: \mathrm{pr}^{-1}(U) \to V_n \,|\, f$ is locally constant in the rigid analytic topology, and $f(\gamma(z)) = \gamma(f(z))$ for $\gamma \in \Gamma\}$. Recall that "locally constant in the rigid analytic topology" means that there exists an admissible cover such that the restriction to each of its members is constant.

The locally free coherent sheaf $\mathcal{O}_X \otimes \boldsymbol{V}_n$ will be denoted by $\mathcal{V}_n$. In contrast to the local system $\boldsymbol{V}_n$, which is irreducible, $\mathcal{V}_1$ fits into a short exact sequence

$$0 \to \omega \to \mathcal{V}_1 \to \omega^{-1} \to 0,$$

where ω is the invertible subsheaf of $\mathcal{V}_1$ defined by the condition that the pull-back of its sections to $\mathfrak{h}_\Gamma$ are of the form $f(z)(u - zv)$, with u, v the canonical basis for V_1 ([dS1], §3.6). It is of degree $g - 1$, $\omega^2 \approx \Omega_{X/K}$, and $H^0(X, \omega^n) = M_n(\Gamma)$ is the space of modular forms of degree n ([dS1] §3.7. Note that there are no cusps or elliptic points.). It follows that $\mathcal{V}_n$ has a descending filtration $\mathcal{V}_n = \mathfrak{F}^0 \supseteq \mathfrak{F}^1 \supseteq \cdots \mathfrak{F}^n \supseteq \mathfrak{F}^{n+1} = 0$ with $\mathfrak{F}^i/\mathfrak{F}^{i+1} \approx \omega^{2i-n}$.

V_n also gives rise to a local system on the tree of Γ (see [dS1]), which we denote by the same letter. The space of Γ-invariant harmonic 1-cocycles on the tree of Γ, with coefficients in the local system V_n, is denoted $C^1_{\mathrm{har}}(V_n)^\Gamma$ (see [dS1] §3.1).

Lemma. *Let $d_n = g$ if $n = 0$, and $(g-1)(n+1)$ if $n \geq 1$. Then*

$$\dim M_{n+2}(\Gamma) = \dim H^1(\Gamma, V_n) = \dim C^1_{\mathrm{har}}(V_n)^\Gamma = d_n.$$

Proof. For $M_{n+2}(\Gamma)$ see [Sch], p. 219. Since Γ is free, $H^1(\Gamma, V_n)$ may be identified with the space of all g-tuples $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_g)$, $\mu_i \in V_n$, modulo those of the form $\mu_i = (\gamma_i - 1)\mu$ for some μ independent of i. Here μ_i is the value of a 1-cocycle representing a specific cohomology class at γ_i. Similarly, $C^1_{\mathrm{har}}(V_n)^\Gamma$ may be identified with the space of all g-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_g)$ such that $\Sigma_i(\gamma_i^{-1} - 1)\lambda_i = 0$. In this identification λ_i is the value of the harmonic 1-cocycle at c_i (we denote by c_i both the annulus, and the oriented edge in the tree of Γ corresponding to it). The dimensions can be calculated using the fact that, since Γ is Zariski dense, if $n \geq 1$, the Γ-invariants or coinvariants of V_n vanish. ∎

There are pairings $H^1(\Gamma, V_n) \times C^1_{\mathrm{har}}(V_n)^\Gamma \to K$ and $C^1_{\mathrm{har}}(V_n)^\Gamma \times H^1(\Gamma, V_n) \to K$ given in this explicit description by $\langle \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle = \Sigma_i \langle \mu_i, \lambda_i \rangle$, and $\langle \boldsymbol{\lambda}, \boldsymbol{\mu} \rangle = \Sigma_i \langle \lambda_i, \mu_i \rangle$. The two pairings differ by the factor $(-1)^n$.

1.3. $H^1_{dR}(X, V_n)$, the (first) de Rham cohomology of X with coefficients in the local system V_n, is defined as the (first) hypercohomology of the coherent sheaf complex $\Omega^*_{X/K} \otimes \mathcal{V}_n$ in the rigid analytic topology,

$$H^1_{dR}(X, V_n) = H^1(X, \Omega^* \otimes \mathcal{V}_n).$$

If $\{U_\alpha\}$ is an admissible covering of X by connected wide open sets (sets isomorphic to the complement of a finite non-empty union of closed disks in $\mathbf{P}^1$), $H^1_{dR}(X, V_n)$ may be computed as Čech cohomology. It is the space of classes $[\{\omega_\alpha\},\{f_{\alpha\beta}\}]$, where $\omega_\alpha \in \Omega \otimes \mathcal{V}_n(U_\alpha)$, $f_{\alpha\beta} \in \mathcal{V}_n(U_{\alpha\beta})$, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = 0$ (where defined), $f_{\alpha\beta} = -f_{\beta\alpha}$, and $\omega_\alpha - \omega_\beta = df_{\alpha\beta}$. The classes are taken modulo families of the form $(\{dg_\alpha\},\{g_\alpha - g_\beta\})$.

We shall use the following basic result.

Theorem (GAGA). *There is a functor from the category of coherent sheaves on X in the Zariski topology, to the category of coherent sheaves on X in the rigid analytic topology, which assigns to $\mathcal{F}$ its analytification $\mathcal{F}^{\mathrm{an}}$. This functor is an equivalence of categories. The cohomology groups $H^q(X,\mathcal{F})$ and $H^q(X,\mathcal{F}^{\mathrm{an}})$ (in the Zariski topology and in the rigid analytic topology) are canonically isomorphic.*

For the proof see [Ko], or [Be], §6.4.

Proposition. *Let η be a V_n-valued differential of the second kind on X (equivalently, a V_n-valued, Γ-invariant differential of the second kind on $\mathfrak{h}_\Gamma$). Let $\{U_\alpha\}$ be a finite wide open cover of X, and on each U_α find a (rigid) meromorphic V_n-valued function g_α such that $\omega_\alpha = \eta - dg_\alpha$ is analytic there. Let $\bar{\eta} = [\{\omega_\alpha\},\{g_\alpha - g_\beta\}] \in H^1_{dR}(X, V_n)$. Then $\eta \mapsto \bar{\eta}$ gives an isomorphism*

$$H^1_{dR}(X, V_n) \approx V_n\text{-valued differentials of the second kind on } X\}/ \\ \{dF \mid F \text{ a } V_n\text{-valued meromorphic function on } X\}.$$

Proof. The only non-obvious point is that every de Rham class is obtained in this way, or that a Čech-1-cocycle $\{f_{\alpha\beta}\}$ may be written in the form $\{g_\alpha - g_\beta\}$ using meromorphic functions (because then $\omega_\alpha + dg_\alpha$ glue). So we have to prove that $H^1(X,\mathcal{M} \otimes \mathcal{V}_n) = 0$, where $\mathcal{M}$ is the sheaf of meromorphic functions. But $\mathcal{M} = \lim \mathcal{L}(D)$ over divisors D, and for any D of a high enough degree, $H^1(X,\mathcal{L}(D) \otimes \mathcal{V}_n) = 0$ by GAGA. ∎

Remark. Unlike the situation in complex geometry, the obvious map from the complex $V_n \to 0$ to the complex $\mathcal{O}_X \to \Omega_X$ is not a quasi-isomorphism, and consequently $H^1(X, V_n)$ is of half the dimension of $H^1_{dR}(X, V_n)$, as we shall see later.

1.4. PROPOSITION ($n > 0$). *There exists a five-term exact sequence*

$$0 \to H^0(X, \mathcal{V}_n) \to H^0(X, \Omega \otimes \mathcal{V}_n) \to H^1_{dR}(X, V_n) \to H^1(X, \mathcal{V}_n) \to H^1(X, \Omega \otimes \mathcal{V}_n) \to 0$$

where the first and last maps are induced from d. Serre duality makes this sequence self-dual. ■

The proof is standard, once we note that $\mathcal{V}_n$ is self-dual. The image of $H^0(X, \Omega \otimes \mathcal{V}_n)$ in $H^1_{dR}(X, V_n)$ *gives the Hodge filtration* $F^1 H^1_{dR}$.

1.5. Let η be a Γ-invariant V_n-valued differential of the second kind on $\mathfrak{h}_\Gamma$. Let $I(\eta)$ be the harmonic 1-cocycle which assigns to an oriented edge e in the tree of Γ the value $I(\eta)(e) = \text{Res}_e(\eta)$. (As before we let e stand for both the edge in the tree and the oriented annulus that it represents.) Let $P_\pi(\eta)$ be the class in $H^1(\Gamma, V_n)$ of the cocycle $\gamma \to \gamma(F_{\pi,\eta}) - F_{\pi,\eta}$. Here $F_{\pi,\eta}$ is Coleman's integral of η in $\mathfrak{h}_\Gamma$, based on Log_π, which is unique up to an additive constant (see [C] and [dS1] §2.3. Log_π is normalized via $\text{Log}_\pi(\pi) = 0$). If η is exact there exists a Γ-invariant V_n-valued meromorphic function F_η in $\mathfrak{h}_\Gamma$ with $dF_\eta = \eta$, so $I(\eta) = 0$, and $P_\pi(\eta) = 0$. This allows us to consider the map ι_π

$$\iota_\pi : H^1_{dR}(X, V_n) \to C^1_{\text{har}}(V_n)^\Gamma \oplus H^1(\Gamma, V_n),$$

$$\iota_\pi : \eta \mapsto (I(\eta), P_\pi(\eta)).$$

1.6. THEOREM (Hodge decomposition). (i) ι_π *is an isomorphism.*

(ii) *Restricted to* $H^{n+1,0} = F^1 H^1_{dR}$, I *is an isomorphism.*

(iii) *Restricted to* $H^{0,n+1} = \text{Ker}(I)$, P_π *is an isomorphism independent of* π, *and the following diagram commutes*:

$$\begin{array}{ccc} H^{0,n+1} & \xrightarrow{P_\pi} & H^1(\Gamma, V_n) \approx H^1(X, V_n) \\ \downarrow & & \downarrow \\ H^1_{dR}(X, V_n) & \longrightarrow & H^1(X, \mathcal{V}_n). \end{array}$$

(iv) *The injection* $M_{n+2}(\Gamma) \to H^0(\Gamma, \Omega_{\mathfrak{h}} \otimes V_n) = H^0(X, \Omega \otimes \mathcal{V}_n)$ (see [dS1]3.8.0) *gives a direct sum decomposition* $H^0(X, \Omega \otimes \mathcal{V}_n) = dH^0(X, \mathcal{V}_n) \oplus M_{n+2}(\Gamma)$.

Differentials of the second kind representing classes in $\text{Ker}(I)$ will be said to have *vanishing annular residues*. We refer to the decomposition $H^1_{dR}(X, V_n) = H^{n+1,0} \oplus H^{0,n+1}$ as the *Hodge decomposition*.

PROOF. Suppose $\iota_\pi(\eta) = 0$. Since all the residues of η on annuli vanish, $F_{\pi,\eta}$ is (rigid) meromorphic. Furthermore, since $P_\pi(\eta) = 0$, we may adjust $F_{\pi,\eta}$ by a constant so that it becomes Γ-invariant. Thus η is exact, proving that ι_π is injective. Now we have shown in theorem 3.9 of [dS1] that I is an isomorphism, when restricted to the image of $M_{n+2}(\Gamma)$ in H^1_{dR}. This shows that $M_{n+2}(\Gamma)$ *injects* into $H^{n+1,0} = F^1H^1_{dR}$. Since by Serre duality $H^1_{dR}/F^1H^1_{dR}$ is dual to $F^1H^1_{dR}$, the dimension of H^1_{dR} is at least $2d_n$. On the other hand we have just seen that it injects, via ι_π, into a $2d_n$-dimensional space. (i), (ii) and (iv) follow immediately. The fact that on $\mathrm{Ker}(I)$ P_π is independent of π follows from Proposition 1.7 below, and the commutativity of the diagram in (iii) results from the definitions. Note that the vertical arrow on the right is induced from the inclusion of V_n in $\mathcal{V}_n$. ■

1.7. PROPOSITION. *Let π' be another uniformizer of K. Then $P_{\pi'}(\eta) - P_\pi(\eta) = \mathrm{Log}(\pi/\pi') \cdot \epsilon \circ I(\eta)$, where ϵ is the map $C^1_{\mathrm{har}}(V_n)^\Gamma \to H^1(\Gamma, V_n)$ obtained from the short exact sequence $0 \to V_n \to C^0_{\mathrm{har}}(V_n) \to C^1_{\mathrm{har}}(V_n) \to 0$ in passing to cohomology.*

PROOF. See proposition 4.2. in [dS1]. The proof there, given for differentials of the first kind, generalizes to differentials of the second kind. Caution: ϵ may not be an isomorphism, but *it is* if Γ is arithmetic or if $n = 0$. ■

REMARK. When ϵ is an isomorphism, for *all but finitely many* π, P_π is an isomorphism also when restricted to $H^{n+1,0}$. Similarly, if $\eta \in \mathrm{Ker}(P_\pi)$ for two π's whose ratio is not a root of unity, then $\eta = 0$.

1.8. CUP PRODUCT. The algebraically defined cup product on $H^1_{dR}(X, V_n)$,

$$[\omega] \cup [\chi] = \Sigma_{z \in \Gamma\backslash\mathfrak{H}} \mathrm{Res}_z \langle F_\omega, \chi \rangle,$$

is given by the following bilinear expression in the periods:

$$[\omega] \cup [\chi] = \langle P_\pi(\omega), I(\chi) \rangle - \langle I(\omega), P_\pi(\chi) \rangle$$

where the pairings between $C^1_{\mathrm{har}}(V_n)^\Gamma$ and $H^1(\Gamma, V_n)$ are the ones described after Lemma 1.2 (see [dS2], theorem 1.6).

COROLLARY. *$H^{0,n+1}$ and $H^{n+1,0}$ are maximal isotropic subspaces under the cup product.* ■

1.9. HODGE–TATE DECOMPOSITION. Recall first the case of trivial coefficients ($n = 0$). We want to show that there exists a canonical decomposition $H^1_{\mathrm{et}}(X_{/\bar{K}}, \mathbf{Z}_p) \otimes \mathbf{C}_p = H^{01} \otimes_K \mathbf{C}_p \oplus H^{10} \otimes_K \mathbf{C}_p(-1)$ (this, of course, is an old result of

Tate and Raynaud). We shall deal, more generally, with an abelian variety A with multiplicative reduction, and we shall adopt the terminology of [dS1], §1.

From the short exact sequence

$$0 \to M \to \mathrm{Hom}(N, \bar{K}^{\times}) \to A(\bar{K}) \to 0$$

we obtain a short exact sequence

$$0 \to \mathrm{Hom}(M,K) \to H^1_{\mathrm{et}}(A_{/\bar{K}}, \mathbf{Z}_p) \otimes K \to N \otimes K(-1) \to 0.$$

By Tate's theorem on the vanishing of $H^1(\mathrm{Gal}(\bar{K}/K), \mathbf{C}_p)$, this sequence will split after tensoring with $\mathbf{C}_p$ over K. It already follows that the etale cohomology is of Hodge–Tate type, in the terminology of Serre [Se]. To complete the picture it remains:

(i) To identify $\mathrm{Hom}(M,K) = H^1(A, \mathcal{O}_A)$. But $H^1(A, \mathcal{O}_A) = t_{A'}(K)$ is the tangent space of the dual abelian variety ([Mu], p. 130), and since $A'(K)$ is the quotient of $\mathrm{Hom}(M, K^{\times})$ by a discrete lattice, its tangent space is canonically identified with $\mathrm{Hom}(M,K)$.

(ii) To identify $N \otimes K = H^0(A, \Omega_{A/K})$. This is a consequence of the analogue of (i) for A', by duality.

(iii) To verify that after the identifications in (i) and (ii) have been made, the maps in the short exact sequence above correspond to the maps constructed by Tate in his paper on p-divisible groups. This is routine, and boils down to the computation of the Weil pairing as in [dS1], 1.5.

Open question ($n > 0$). Let $H^1_{\text{rig-et}}(X_{/\bar{K}}, V_n)$ denote the rigid-etale cohomology with coefficients in the local system V_n. Note that this *cannot* be identified with the usual etale cohomology for any l.c.c. sheaf, because V_n is not obtained from the projective limit of modules on which Γ acts through finite quotients, since it does not have a Γ-invariant lattice in it. Nevertheless, $H^1_{\text{rig-et}}(X_{/\bar{K}}, V_n)$ is a finite dimensional K-vector space with $\mathrm{Gal}(\bar{K}/K)$ action. Prove that it is of Hodge–Tate type, and identify the factors in its decomposition with Tate-twists of $H^{n+1,0}$ and $H^{0,n+1}$. The weights should be $-n-1$ and 0.

§2. The differential equations satisfied by the differentials of the second kind in families of Mumford curves

2.1. In §2 we shall look at the Hodge decomposition in a family of Mumford curves over a base S. When $n = 0$ (so the Gauss–Manin connection is defined) we shall give a new proof of Gerritzen's theorem, that it constitutes a *variation of Hodge structure*.

Let S be a reduced connected rigid analytic space over K, $\Gamma \subseteq \mathrm{Aut}_S(\mathbf{P}^1 \times S)$ a *Schottky group over* S (see [Ge3] §5), and $Z \subseteq \mathbf{P}^1 \times S$ the domain of ordinary points of Γ ([Ge3], proposition 5). Since all our results will be of local nature on the base (in the rigid topology) we assume that $S = \mathrm{Sp}\, A$ is an affinoid. For each geometric point $s \in S$, $\Gamma_s \subseteq \mathrm{PGL}_2(\mathbf{C}_p)$ is the Schottky group obtained by specialization. $X = \Gamma \backslash Z$ is a rigid analytic family of Mumford curves, and the fiber above s we denote by X_s. We need a result on the structure of a fundamental domain for Γ, which is the relative version of Proposition 1.1.

2.2. Proposition. *Possibly after replacing S by a finite cover by affinoids, there exist* $\gamma_i,\ \alpha_i \in \mathrm{GL}_2(\mathcal{O}_S(S))$, *and* $\mu_i \in \mathcal{O}_S(S)^\times$, $1 \le i \le g$, $\|\mu_i\|_{\sup} < 1$, *with the following properties*:

(i) *At each $s \in S$, the classes of $\gamma_i(s)$ in* PGL_2 *form a Schottky basis for the group* Γ_s.

(ii) $\alpha_i(s)\gamma_i(s)\alpha_i(s)^{-1}(z) = \mu_i(s)\cdot z$.

(iii) *For any $s \in S$, and $i \neq j$, the fixed points of $\gamma_j(s)$ lie in* $W_i(s) = \{z \mid |\mu_i(s)| < |\alpha_i(s)(z)| \le 1\}$.

Let $W_i = \{(z,s) \in \mathbf{P}^1 \times S \mid (\in W_i(s)\}$, *and* $W = \cap W_i$. *Then W is contained in Z, and is a fundamental domain for* $\Gamma\backslash Z$.

Proof. The proposition follows from the proof of Proposition 6 in [Ge3] by an argument similar to the one used in the proof of Proposition 1.1 above. ∎

Let $z_j^{\pm}(s)$ be the two fixed points of $\gamma_j(s)$, $s \in S$. Since $\alpha_i(s)(z_j^{\pm}(s))/\mu_i(s)$ $(i \neq j)$ is a rigid analytic function on the affinoid S which is everywhere bigger than 1 in absolute value, by the maximum modulus principle it is bigger than some $r > 1$, $r \in |\mathbf{C}_p^\times|$. It follows that the "family of annuli" (see [BGR] §9.7.1)

$$c_{i,r} = \{(z,s) \mid |\mu_i(s)| < |\alpha_i(s)(z)| < r|\mu_i(s)|\}$$

is contained in W. Similarly define annuli $b_{i,r} = \gamma_i^{-1}(c_{i,r})$ (compare with §1.1).

2.3. We shall assume that either n is even, or (at least after replacing S by a finite cover by affinoids) the $\gamma_i(s)$ in Lemma 2.2 can be chosen in $\mathrm{SL}_2(\mathcal{O}_S(S))$. Under this assumption $V_n(\mathbf{C}_p)$ supplies a family of representations for the groups Γ_s, or, equivalently, a (holomorphically varying) family of local systems $\mathbf{V}_n$ on the curves X_s. If pr denotes the projection from $Z \subseteq \mathbf{P}^1 \times S$ to X, then for any open admissible $U \subseteq X$,

$$\mathbf{V}_n(U) = \Big\{ f = f(z,s) : \mathrm{pr}^{-1}(U) \to V_n \,\Big|\, f \text{ is rigid analytic, for each } s \in S\ f(z,s) \text{ is locally constant on } \mathrm{pr}^{-1}(U) \cap (\mathbf{P}^1 \times \{s\}) \text{ as a function of } z, \text{ and for each } \gamma \in \Gamma,\ \gamma \circ f \circ \gamma^{-1} = f \Big\}.$$

Obviously this is a sheaf of $\mathcal{O}_S$-modules. The coherent sheaf $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathbf{V}_n$ is denoted by $\mathcal{V}_n$.

2.4. Denote by π and $\tilde{\pi}$ the projections from X and Z to S, so that $\tilde{\pi} = \pi \circ \mathrm{pr}$. The (first) relative de Rham cohomology with coefficients in the local system $\mathbf{V}_n$ is defined as the (first) hypercohomology of the coherent sheaf complex $\Omega^*_{X/S} \otimes \mathcal{V}_n$ in the rigid analytic topology.

$$\mathcal{H}^1_{dR}(X,\mathbf{V}_n) = \mathbf{R}^1\pi_*(\Omega^* \otimes \mathcal{V}_n).$$

In practice it is computed as Čech hypercohomology using a finite wide open cover of X. The Hodge filtration in this case is given by

$$\mathcal{H}^{n+1,0} = \pi_*(\Omega^* \otimes \mathcal{V}_n)/d(\pi_*\mathcal{V}_n) \subseteq \mathcal{H}^1_{dR}(X,\mathbf{V}_n).$$

PROPOSITION. *$\mathcal{H}^{n+1,0}$ and $\mathcal{H}^1_{dR}(X,\mathbf{V}_n)$ are coherent locally free $\mathcal{O}_S$-modules of ranks d_n and $2d_n$ respectively. Their fibers at a point $s \in S$ are identified with $H^{n+1,0}(X_s)$ and $H^1_{dR}(X_s,\mathbf{V}_n)$.*

PROOF. Coherence follows from Kiehl's theorem, since π is proper. Now $\Omega^* \otimes \mathcal{V}_n$ is flat over $\mathcal{O}_S$, S is connected and reduced, and the dimensions of the cohomologies of the fibers of π are *constant.* The proposition follows by the rigid analytic analogue of a well-known consequence of the semicontinuity theorem in algebraic geometry, which allows one to replace the fiber of the cohomology by the cohomology of the fiber under these conditions. (See [Mu], §5 for the theorem used here, and [Be], corollary 5.5.9 for the rigid analytic analogue. Granted Kiehl's theorem, the deduction of the semicontinuity theorem and its corollaries are, *mutatis mutandis,* the same as in Mumford's book. The extension of the theorem from cohomology to hypercohomology may be justified, for example, using the "Hodge to de Rham" spectral sequence.) ∎

The proposition tells us that sections of $\mathcal{H}^1_{dR}(X,\mathbf{V}_n)$ may be identified with $\mathbf{V}_n$-valued relative differentials on X/S, all of whose specializations to fibers of π are of the second kind (we shall call such a differential a relative differential of the second kind), modulo exact differentials. More precisely,

$$\mathcal{H}^1_{dR}(X,\mathbf{V}_n) = \{f(z,s)dz \mid f \text{ is } V_n\text{-valued meromorphic in } Z \subseteq \mathbf{P}^1 \times S, \text{ for fixed } s\ f(z,s)dz \text{ is of the second kind in its domain of definition, and } \gamma \circ f = f \circ \gamma \text{ for all } \gamma \in \Gamma\} / \{d_z F(z,s) \mid F \text{ is } \Gamma\text{-invariant, } V_n\text{-valued and meromorphic in } Z\}.$$

2.5. We shall now show that the complements $H^{0,n+1}(X_s)$ to the space of V_n-valued differentials of the first kind on X_s patch together to give a decomposition of locally free coherent sheaves

$$\mathcal{H}^1_{dR}(X,\mathbf{V}_n) = \mathcal{H}^{n+1,0} \oplus \mathcal{H}^{0,n+1}.$$

Let $\tilde{\pi}$ be the projection of Z to S, and $\mathfrak{M}_Z$ the (quasi-coherent) module of meromorphic, V_n-valued functions on Z. Consider the homomorphism of $\mathcal{O}_S$-modules

$$d_{Z/S} : \tilde{\pi}_* \mathfrak{M}_Z \to \tilde{\pi}_* (\Omega_{Z/S} \otimes_{\mathcal{O}_Z} \mathfrak{M}_Z).$$

Let $\mathfrak{D}$ be the image sheaf, again a quasi-coherent sheaf on S. The group Γ acts on the cover $Z \to S$ as a discontinuous group of rigid analytic automorphisms, and the action is free. It also acts on the coefficients (V_n), hence induces an action on the sheaves $\tilde{\pi}_* \mathfrak{M}_Z$ and $\tilde{\pi}_* (\Omega_{Z/S} \otimes_{\mathcal{O}_Z} \mathfrak{M}_Z)$ which commutes with $d_{Z/S}$. It follows that Γ acts on $\mathfrak{D}$, and the Γ-invariants of $\mathfrak{D}$, denoted $\mathfrak{D}^\Gamma$, is again a quasi-coherent sheaf on S. Now the sections of $\mathfrak{D}^\Gamma$ are just the relative $\mathbf{V}_n$-valued differentials of the second kind on X/S, whose restriction to each fiber X_s has vanishing annular residues in the sense of §1. Let $\mathcal{E}$ be the sheaf $d_{X/S}\pi_*\mathfrak{M}_X$ of relative $\mathbf{V}_n$-valued exact differentials on X/S ($\pi_*\mathfrak{M}_X = \tilde{\pi}_*\mathfrak{M}_Z^\Gamma$). Clearly $\mathcal{E} \subseteq \mathfrak{D}^\Gamma$. Recall that the relative de Rham cohomology, $\mathcal{H}^1_{dR}(X/S,\mathbf{V}_n)$ may be identified with the sheaf of relative $\mathbf{V}_n$-valued differentials of the second kind on X/S, modulo exact differentials. This discussion leads to the following.

Proposition. *The inclusion $\mathcal{H}^{0,n+1} := \mathfrak{D}^\Gamma/\mathcal{E} \subseteq \mathcal{H}^1_{dR}(X/S,\mathbf{V}_n)$ is an inclusion of coherent sheaves of $\mathcal{O}_S$-modules. At each point s of S it yields, by passing to the fiber, the inclusion $H^{0,n+1}(X_s) \subseteq H^1_{dR}(X_s,\mathbf{V}_n)$* (see Theorem 1.5). *We therefore have a direct sum decomposition of coherent locally free sheaves*

$$\mathcal{H}^1_{dR}(X/S,\mathbf{V}_n) = \mathcal{H}^{n+1,0} \oplus \mathcal{H}^{0,n+1}$$

Proof. $\mathfrak{D}^\Gamma/\mathcal{E}$, as a submodule of a locally free coherent module, is again coherent and locally free, and at any s it specializes to a subspace of $H^{0,n+1}(X_s)$ by definition. The sum on the right is therefore direct. To prove the proposition it remains to show that, possibly after passing to a finite cover of S by affinoids, every $\mathbf{V}_n$-valued differential of the second kind with vanishing annular residues on a single fiber X_0 extends to a relative $\mathbf{V}_n$-valued differential of the second kind on X/S, all of whose specializations have vanishing annular residues.

We may assume that X has a fundamental domain as described in §2.2, and that $\pi_*(\Omega_{X/S} \otimes \mathcal{V}_n)/d(\pi_*\mathcal{V}_n) = \mathcal{H}^{n+1,0}$ is free with a basis ω_i over $\mathcal{O}_S$. Let η_0 be a $\mathbf{V}_n$-valued differential of the second kind on X_0, with vanishing annular residues. By Proposition 2.4 we may assume (maybe after replacing S by a finite affinoid cover) that η_0 is obtained by specialization at s_0, from a relative differential of the second kind η.

Now $\mathrm{Res}_{c_{i,r}(s)}\eta_s$ is a rigid analytic function of s ($c_{i,r}(s)$ is the annulus defined in §2.2). This follows from a "Mittag-Leffler decomposition over S" for rigid analytic functions in W, which in the case $g = 1$ may be found in [BGR] 9.7.1, and in general is proved in Lemma 2.6 below. (Over $\mathbf{C}$ such a theorem follows from Cauchy's integral formula.) So are the functions $\mathrm{Res}_{c_{i,r}(s)}\omega_k$ for every i and k. Identify $C^1_{\mathrm{har}}(V_n)^{\Gamma_s}$ with the space of g-tuples $\boldsymbol{\lambda}$ as in Lemma 1.2, the annuli $c_{i,r}(s)$ taking the place of c_i. Thus $I(\eta_s) = (\ldots, \mathrm{Res}_{c_{i,r}(s)}\eta_s, \ldots)$ etc. As $I((\omega_k)_s)$ form a basis for $C^1_{\mathrm{har}}(V_n)^{\Gamma_s}$ by Theorem 1.6, there exist rigid analytic functions h_k on S such that $\eta - \Sigma h_k\omega_k$ is in the kernel of I at each s. This is the desired modification of η that lies in $\mathfrak{D}^\Gamma$. ■

2.6. LEMMA (Mittag-Leffler decomposition over S). *Let W be the space defined in Proposition 2.2. Then possibly after replacing S by a finite affinoid cover, every rigid analytic function h in W has a decomposition*

$$h = \sum_{1 \le i \le g} h_i$$

where h_i is rigid analytic in W_i. Each $h_i = h_i' + h_i''$, where h_i' (resp. h_i'') is analytic in $\{(z,s) \mid |\mu_i(s)| < |\alpha_i(s)(z)|\}$ (resp. in $\{(z,s) \mid |\alpha_i(s)(z)| \le 1\}$).

PROOF. For r small enough consider the restriction of h to the "annulus" $c_{i,r}$. Making a global change of the z-coordinate we may assume that $\alpha_i(s) = \mu_i(s) = 1$. Clearly then we may find h_i' and g_i' of the form

$$h_i' = \sum_{k<0} h_{i,k}'(s)z^k, \qquad g_i' = \sum_{k \ge 0} g_{i,k}'(s)z^k$$

converging in $|z| > 1$ and in $|z| < r$ respectively, whose sum is $h|c_{i,r}$. It follows that $h - h_i'$ extends to $W \cup \{(z,s) \mid |\alpha_i(s)(z)| \le |\mu_i(s)|\}$. Similarly find h_i'' so that $h - h_i' - h_i''$ extends to $\bigcap_{j \ne i} W_j$. The lemma is now proved by induction. ■

2.7. THE GAUSS–MANIN CONNECTION. From now on $n = 0$. The Gauss–Manin connection

$$\nabla : \mathcal{H}^1_{dR}(X/S) \to \mathcal{H}^1_{dR}(X/S) \otimes \Omega_S$$

is defined as follows. Let D be any derivation (≡vector field) in S/K, i.e. a global section of the dual of Ω_S. Let $\bar{\eta}$ be a global section of $\mathcal{H}^1_{dR}(X/S)$. We shall describe the contraction of $\nabla\bar{\eta}$ with D, which we denote $\nabla_D(\bar{\eta})$. Let $\{U_\alpha\}$ be an admissible cover of X, such that in each U_α there exists a rigid analytic function x_α with the property that U_α is étale over Sp $A\langle\langle x_\alpha\rangle\rangle$. This is possible since X is smooth of relative dimension 1 over $S = \mathrm{Sp}\, A$. Clearly $d_{X/S}x_\alpha$ generates the module of relative differentials on U_α.

Now $\bar{\eta} = [\{\eta_\alpha\}, \{g_{\alpha\beta}\}]$, where $\eta_\alpha \in \Omega_{X/S}(U_\alpha)$, $g_{\alpha\beta} \in \mathcal{O}_X(U_{\alpha\beta})$ and $dg_{\alpha\beta} = \eta_\alpha - \eta_\beta$ ($d = d_{X/S}$). Write $\eta_\alpha = F_\alpha dx_\alpha$. Then on $U_{\alpha\beta}$

$$F_\alpha = F_\beta(\partial x_\beta/\partial x_\alpha) + \partial g_{\alpha\beta}/\partial x_\alpha.$$

Extend D to U_α so that it acts trivially on x_α and call the extension D_α. In $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have the relation

$$D_\alpha - D_\beta = (D_\alpha x_\beta)\partial/\partial x_\beta$$

that holds because both sides agree on A and on x_β. Define

$$\xi_\alpha = (D_\alpha F_\alpha)dx_\alpha,$$

$$h_{\alpha\beta} = (D_\alpha x_\beta)F_\beta + D_\alpha g_{\alpha\beta}.$$

Then $\xi_\alpha - \xi_\beta = dh_{\alpha\beta}$, and we set $\nabla_D(\bar{\eta}) = \bar{\xi} = [\{\xi_\alpha\}, \{h_{\alpha\beta}\}]$. It is easy to check that this is well defined.

2.8. Let η be a relative differential of the second kind representing $\bar{\eta}$ (see the remark after Proposition 2.4). Then $\eta_\alpha = \eta + dg_\alpha$ for g_α meromorphic in U_α, and $g_{\alpha\beta} = g_\alpha - g_\beta \in \mathcal{O}_X(U_{\alpha\beta})$.

If furthermore $\bar{\eta} \in \mathcal{H}^{01}$, then $P(\eta_s) = P_\pi(\eta_s) \in \mathrm{Hom}(\Gamma_s, \mathbf{C}_p)$ is independent of π (Theorem 1.5(iii)). Since Γ is isomorphic to Γ_s under specialization at s, we may view $P(\eta_s)$ as a function $S \to \mathrm{Hom}(\Gamma, \mathbf{C}_p)$, which we simply denote by $P(\eta)$.

THEOREM. *Possibly after replacing S by a finite affinoid cover, there exists a basis* $\{\bar{\eta}_k\}$ *of* $\mathcal{H}^{01}$ *for which the functions* $P(\eta_k)$ *are constant. Such a basis is horizontal for the Gauss–Manin connection.*

PROOF. The proof of the first part is similar to that of Proposition 2.5. We may assume that a Schottky basis $\gamma_i(s)$ as in Lemma 2.2 is fixed, and identify

$\mathrm{Hom}(\Gamma,\mathbf{C}_p)$ with the space of g-tuples $\boldsymbol{\mu}$ as in Lemma 1.2. We may further assume that a basis $\bar{\eta}_k$ of $\mathcal{H}^{01}$ is given, and that $\mathrm{pr}^*\eta_k = d_{z/s}F_k$ for a rigid meromorphic function F_k on Z. This follows from the definition of $\mathfrak{D}$. Now

$$P(\eta_k)(s) = (\ldots, F_k(\gamma_i(s)^{-1}(z),s) - F_k(z,s), \ldots),$$

so the entries are rigid analytic in s. Since P is an isomorphism of H^{01} onto $\mathrm{Hom}(\Gamma,\mathbf{C}_p)$, the matrix expressing $P(\eta_k)$ in terms of a fixed basis of $\mathrm{Hom}(\Gamma,\mathbf{C}_p)$ has rigid analytic entries and non-vanishing determinant. Inverting the matrix and applying it to the basis $\{\eta_k\}$ we get another basis on which the map P is constant.

Assume now that η is a relative differential of the second kind representing a class in $\mathcal{H}^{01}$, and that the period map $P(\eta)$ is constant. Keep the notation used in the definition of the Gauss–Manin connection above. We may assume that the cover $\{U_\alpha\}$ by connected affinoid subdomains satisfies the following assumptions:

(i) There exists a $\tilde{U}_\alpha \subseteq Z$ isomorphic to U_α under pr,

(ii) $\mathrm{pr}^{-1}(U_\alpha) = \cup_{\gamma\in\Gamma}\gamma(\tilde{U}_\alpha)$, a disjoint union,

(iii) for each pair of indices α and β there exists a unique $\gamma \in \Gamma$ such that pr induces an isomorphism from $\tilde{U}_\beta \cap \gamma(\tilde{U}_\alpha)$ onto $U_\beta \cap U_\alpha$.

Let $p_\alpha = \mathrm{pr}\,|\,\tilde{U}_\alpha$, and $p_{\alpha,\gamma} = p_\alpha \circ \gamma^{-1}$. These isomorphisms induce vector fields $\tilde{D}_\alpha$ and $\tilde{D}_{\alpha,\gamma}$ on $\tilde{U}_\alpha$ and $\gamma(\tilde{U}_\alpha)$ that correspond to D_α on U_α. Now $\eta \circ \mathrm{pr} = dF$ for a meromorphic function F in Z $(d = d_{Z/S})$. Define

$$\tilde{u}_\alpha = \tilde{D}_\alpha(F + \tilde{g}_\alpha) \quad (\tilde{g}_\alpha = g_\alpha \circ p_\alpha), \qquad u_\alpha = \tilde{u}_\alpha \circ p_\alpha^{-1}.$$

We shall show: $\xi_\alpha = d_{X/S}u_\alpha$, and $h_{\alpha\beta} = u_\alpha - u_\beta$. This will imply $\nabla_D(\bar{\eta}) = 0$.

(a) Let us compute

$$d_{Z/S}\tilde{u}_\alpha = \tilde{D}_\alpha(\partial(F + \tilde{g}_\alpha)/\partial\tilde{x}_\alpha)\cdot d\tilde{x}_\alpha = \tilde{D}_\alpha\tilde{F}_\alpha \cdot d\tilde{x}_\alpha = (D_\alpha F_\alpha \cdot dx_\alpha)\circ p_\alpha = \xi_\alpha \circ p_\alpha.$$

The first assertion follows from here. Note that in the first equality we used the fact that D_α and $\partial/\partial x_\alpha$ commute.

(b) To check that $h_{\alpha\beta} = u_\alpha - u_\beta$ we denote by p the projection from $\tilde{U}_\beta \cap \gamma(\tilde{U}_\alpha)$ to $U_\alpha \cap U_\beta$, and by $\tilde{}$ the pull back of functions or differentials via p. Now

$$\begin{aligned}\tilde{h}_{\alpha\beta} &= (\tilde{D}_{\alpha,\gamma}\tilde{x}_\beta)\cdot\tilde{F}_\beta + \tilde{D}_{\alpha,\gamma}(\tilde{g}_\alpha - \tilde{g}_\beta)\\ &= (\tilde{D}_{\alpha,\gamma}\tilde{x}_\beta)\cdot\partial/\partial\tilde{x}_\beta(F + \tilde{g}_\beta) + \tilde{D}_{\alpha,\gamma}(\tilde{g}_\alpha - \tilde{g}_\beta)\\ &= (\tilde{D}_{\alpha,\gamma} - \tilde{D}_\beta)(F + \tilde{g}_\beta) + \tilde{D}_{\alpha,\gamma}(\tilde{g}_\alpha - \tilde{g}_\beta).\end{aligned}$$

On the other hand $\tilde{u}_\alpha - \tilde{u}_\beta = (\tilde{D}_\alpha(F + g_\alpha \circ p_\alpha))\circ\gamma^{-1} - \tilde{D}_\beta(F + \tilde{g}_\beta)$. Thus the difference between $\tilde{h}_{\alpha\beta}$ and $\tilde{u}_\alpha - \tilde{u}_\beta$ is $\tilde{D}_{\alpha,\gamma}F - (\tilde{D}_\alpha F)\circ\gamma^{-1} = \tilde{D}_\alpha(F\circ\gamma - F)\circ\gamma^{-1} =$

$D(F\circ\gamma - F)$, which vanishes since $F\circ\gamma - F = P(\eta)(s)(\gamma)$ is a constant function of s. ■

2.9. Fix a fundamental domain for $\Gamma\backslash Z$ and a Schottky basis for Γ as in Proposition 2.2. Let $\{\bar{\eta}_k\}$ be a basis of $\mathcal{H}^{01}$ as in the last theorem, satisfying $P(\eta_k)(\gamma_j) = \delta_{kj}$.

THEOREM. (i) *There exists a basis* $\{\omega_i\}$ *of* $\mathcal{H}^{10}$ *for which* $I(\omega_i)$ *are constant. If* $I(\omega)$ *is constant then* $d_S(P_\pi(\omega)(\gamma_k))$ *is rigid analytic and independent of* π.

(ii) *If* $I(\omega)$ *is constant, then*

$$\nabla\omega = \sum_{1\le k\le g} \eta_k \otimes d_S(P_\pi(\omega)(\gamma_k)).$$

In particular, $\nabla\omega \in \mathcal{H}^{01}\otimes\Omega_S$.

PROOF. We shall not prove part (i), which is the easy half of the theorem. The existence of a basis as required is proven in the same way as in Theorem 2.8. The second statement is proven with the aid of the relative Mittag-Leffler decomposition (Lemma 2.6).

For (ii) let D be a derivation on S/K, and $\bar{\xi} = \nabla_D(\bar{\omega})$, where $\bar{\omega}$ is the class of ω. Let ξ be a differential of the second kind representing $\bar{\xi}$. We have to check

(a) $I(\bar{\xi}) = 0$,

(b) $P_\pi(\xi)(\gamma_j) = D(P_\pi(\omega)(\gamma_j))$, $\forall 1\le j\le g$.

The proof of (a) is relatively straightforward: Let $\partial/\partial s$ denote the pull-back of D to Z via $\tilde{\pi}$. Notation as in §2.7, we also use $\tilde{}$ to denote pull-back from X to Z, e.g. $\tilde{\omega} = \omega\circ\mathrm{pr}$, and $\tilde{D}_\alpha = (\mathrm{pr}^{-1})_* D_\alpha$. Now the relation $\tilde{D}_\alpha - \partial/\partial s = (\tilde{D}_\alpha z)\cdot\partial/\partial z$, which holds in Z, implies

$$\tilde{\xi}_\alpha - \partial/\partial s(\tilde{\omega}) = d_{Z/S}((\tilde{D}_\alpha z)\cdot(\tilde{\omega}/dz)),$$

so in particular we may replace $\tilde{\xi}$ by $\partial/\partial s(\tilde{\omega})$ when we compute its residues along annuli (albeit the latter is not Γ invariant). However, it is easy to see, using the Mittag-Leffler decomposition of $\tilde{\omega}$, and the fact that its residues are constant functions of s, that the residues of $\partial/\partial s(\tilde{\omega})$ all vanish. This proves (a).

For the proof of (b) choose a rational function x on X, and let D_x be the extension of D to a derivation of the function field of X which vanishes on x. Write $\omega = H\cdot dx$, and observe that for a differential of the second kind representing $\bar{\xi}$ we may take $\xi = (D_x H)\cdot dx$. (Incidentally, we have preferred Manin's original definition of the Gauss–Manin connection to the one given in §2.7. This is because the interpretation of the de Rham cohomology by means of differentials of the second kind is more natural, for computations involving P_π, than the Čech interpretation.)

The right-hand side of (b) is

(1) $$\partial/\partial s(F_{\pi,\omega}(\gamma_j(s)^{-1}(z),s) - F_{\pi,\omega}(z,s)).$$

The left-hand side is computed as

(2) $$G(\gamma_j(s)^{-1}(z),s) - G(z,s)$$

where G is any rigid meromorphic function in Z such that

(3) $$\partial G/\partial z = (\tilde{D}_x H)\cdot(\partial\tilde{x}/\partial z).$$

Claim. We may take

(4) $$G = \partial F_{\pi,\omega}/\partial s - (\partial F_{\pi,\omega}/\partial z)\cdot(\partial\tilde{x}/\partial s)/(\partial\tilde{x}/\partial z).$$

To justify the claim, compute the partial derivative of G with respect to z, using the following relation:

(5) $$\tilde{D}_x - \partial/\partial s = (\tilde{D}_x z)\cdot\partial/\partial z.$$

Finally, the equality (1) = (2) boils down, after some computations, to the statement that $\tilde{x}$ is invariant under Γ, simply because it is the pull-back of a function from X. This concludes the proof of the theorem. ■

Theorems 2.8 and 2.9 together are equivalent to the statement that the decomposition of $\mathcal{H}^1_{dR}$ studied in §2 of this paper is a *variation of Hodge structure* (when $n = 0$, otherwise the Gauss-Manin connection is not defined). In fact, there is a polarization too. If η is as in 2.8, and ω as in 2.9, the formula in §1.8 shows that their cup product is constant on S.

Theorems 2.8 and 2.9 are a restatement of the main theorem of Gerritzen's paper [Ge3]. His proof, however, was carried on the Jacobian of X, and employed explicit formulas for differentials of the first kind derived from theta functions on the Jacobian.

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